Quasi-stationary states and incomplete violent relaxation in systems with long-range interactions

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Abstract

We discuss the nature of quasi-stationary states (QSS) with non-Boltzmannian distribution in systems with long-range interactions in relation with a process of incomplete violent relaxation based on the Vlasov equation. We discuss several attempts to characterize these QSS. We show that their distribution is *non-universal* and explain why their prediction is difficult in general.

Key words: Vlasov equation, long-range interactions, quasi-stationary states

1 Quasi-stationary states: a generalized thermodynamics?

It has been observed in many domains of physics [1] that Hamiltonian systems with long-range interactions spontaneously organize into coherent structures which persist for a long time. Some examples are galaxies in astrophysics, jets and vortices in 2D geophysical flows, clusters in the HMF model etc. These quasi-stationary states (QSS) are usually *not* described by the Boltzmann distribution. To account for this striking observational fact, some authors have proposed to replace the Boltzmann entropy by the Tsallis entropy

$$S_q[f] = -\frac{1}{q-1}\int (f^q - f)\, d\mathbf{r} d\mathbf{v}. \tag{1}$$

The reason advocated is that the system is non-extensive so that standard thermodynamics may not be applicable [2]. The maximization of the Tsallis entropy at fixed mass and energy leads to q-distributions of the form $f(\mathbf{r},\mathbf{v}) = [\mu - \beta(q-1)\epsilon/q]^{\frac{1}{q-1}}$, where $\epsilon = v^2/2 + \Phi(\mathbf{r})$ is the individual energy and μ, β are Lagrange multipliers. There are situations where these distributions provide a good *fit* [2,3] of the QSS. However, there exists other situations [4] that are described neither by the Boltzmann nor by the Tsallis distribution [5]. We want to show that the prediction of the QSS is relatively complicated and explain why. To that purpose, we use classical methods of kinetic theory relying on the Vlasov equation [1].

2 Kinetic theory: the importance of the Vlasov equation

To understand the physics of the problem, we first have to develop a kinetic theory of systems with long-range interactions [1,6]. Such kinetic theories have been developed for stellar systems, two-dimensional vortices, the HMF model etc. They usually lead to a kinetic equation for the distribution function $f(\mathbf{r}, \mathbf{v}, t)$ of the form

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla\Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{1}{N^{\delta}} Q(f). \tag{2}$$

The left hand side, called the Vlasov term, describes an advection in phase space due to the mean-field potential $\Phi(\mathbf{r}, t) = \int u(|\mathbf{r} - \mathbf{r}'|) f(\mathbf{r}', \mathbf{v}', t) d\mathbf{v}' d\mathbf{r}'$ where u is a binary potential of interaction. The right hand side takes into account the effect of "collisions" (more generally correlations) between particles and depends on the number N of particles. For long-range interactions, it usually scales like $N^{-\delta}$ with $\delta \geq 1$ (in general $\delta = 1$ but for 1D systems $\delta > 1$). This term, due to finite N effects (graininess), is responsible for the *collisional relaxation* of the system towards ordinary statistical equilibrium. Indeed, in general, the collision term cancels out only for the Boltzmann distribution: $Q(f_e) = 0 \leftrightarrow f_e = Ae^{-\beta m \epsilon}$. However, due to its dependence with the number of particles, the collisional relaxation time is of order $t_R \sim N^{\delta} t_D$ where t_D is the dynamical time. In general, this timescale is huge and does not represent the regime of most physical interest. In particular, the QSS mentioned previously form on a timescale $t \sim t_D \ll t_R$ for which the collision term can be neglected in a first approximation. In that case, the evolution is collisionless and described by the Vlasov equation: $df/dt = 0$. Now, it has been understood, first in astrophysics by Hénon, King and Lynden-Bell in the 1960's, that the Vlasov equation, when coupled to a long-range force like the gravitational force (Vlasov-Poisson system) can undergo a form of *collisionless relaxation* on a very short timescale $\sim t_D$. This is called *violent relaxation* [7]. This general process explains the ubiquity of long-lived QSS in systems with long-range interactions. The fine-grained DF $f(\mathbf{r}, \mathbf{v}, t)$ which is solution of the Vlasov equation never achieves equilibrium but develops intermingled filaments at smaller and smaller scales due to *phase mixing*. However, if we locally average over these filaments, the coarse-grained DF $\overline{f}(\mathbf{r}, \mathbf{v}, t)$ is expected to converge toward a steady state which is a stable stationary solution of the Vlasov equation. This is called weak convergence in mathematics. The kinetic theory explains that the lifetime of the QSS scales as a power N^{δ} of the number of particles. The kinetic theory also explains that when the $t \to +\infty$ limit is taken before the $N \to +\infty$ limit we get the Boltzmann statistical equilibrium state (collisional relaxation) but when the $N \to +\infty$ limit is taken before the $t \to +\infty$ limit we get a QSS which is a stable stationary solution of the Vlasov equation usually different from the Boltzmann distribution (collisionless regime). This non-commutation of the limits $N \to +\infty$ and $t \to +\infty$ has been observed numerically by [2] although they did not give an explanation in terms of the Vlasov equation, as was done in [1] (see Sec. 6.5) but rather in terms of Tsallis generalized thermodynamics. On short timescales $\sim t_D$, the system undergoes violent relaxation and reaches a stationary solution of the Vlasov equation (on the coarse-grained scale). Since the Vlasov equation admits an infinite number of stationary solutions, the solution (QSS) effectively selected by the evolution is difficult to predict (see below). On intermediate timescales $t_D < t < t_R$, the system passes by a succession of quasi-stationary states that are quasi-stationary solutions of the Vlasov equation $f(\epsilon, t)$ slowly evolving with time due to "collisions" (finite N effects). In astrophysics, this slow collisional evolution is governed by the orbit-averaged-Fokker-Planck equation. On a longer timescale $\sim t_R$, collisions finally select the Maxwell distribution among all stationary solutions of the Vlasov equation (the fate of gravitational systems is peculiar due to the absence of strict statistical equilibrium and the process of evaporation or collapse). These different regimes have been observed for different physical systems [1,6] and are illustrated numerically by Yamaguchi *et al.* [8] for the HMF model.

3 Lynden-Bell's theory of violent relaxation

In a seminal paper, Lynden-Bell [7] tried to *predict* the QSS resulting from violent relaxation assuming that the system mixes well (ergodicity) and using arguments of statistical mechanics. However, the statistical mechanics of the Vlasov equation is peculiar because of the presence of an infinite set of constraints: the Casimirs $I_h = \int h(f) d\mathbf{r} d\mathbf{v}$ (for any function h) which contain all the moments $I_n = \int f^n d\mathbf{r} d\mathbf{v}$ of the distribution function. These are sort of "hidden constraints" [5] because they are not accessible from the (observed) coarse-grained DF $\overline{f}$ since $\int \overline{f^n} d\mathbf{r} d\mathbf{v} \neq \int \overline{f}^n d\mathbf{r} d\mathbf{v}$ for $n \neq 1$. Therefore, the proper density probability to consider in the statistical theory of violent relaxation is $\rho(\mathbf{r}, \mathbf{v}, \eta)$ which gives the density probability of finding the level of DF $f = \eta$ in $(\mathbf{r}, \mathbf{v})$ in phase space. Note that we make the statistical mechanics of a *field*, the distribution function, not the statistical mechanics of *discrete particles*. From this density probability, we can construct all the coarse-grained moments $\overline{f^n} = \int \rho \eta^n d\eta$ including the coarse-grained DF $\overline{f} = \int \rho \eta d\eta$. We can now introduce a mixing entropy from a combinatorial analysis [7,5] like in Boltzmann's traditional approach. In the context of the Vlasov equation, this entropy is a functional of $\rho(\mathbf{r},\mathbf{v},\eta)$ of the form

$$S_{L.B.}[\rho] = -\int \rho(\mathbf{r},\mathbf{v},\eta)\ln\rho(\mathbf{r},\mathbf{v},\eta)d\mathbf{r}d\mathbf{v}d\eta. \tag{3}$$

The Lynden-Bell entropy (3) is the proper form of Boltzmann entropy taking into account the specificities of the Vlasov equation. Assuming ergodicity (efficient mixing), the QSS is obtained by maximizing $S_{L.B.}[\rho]$ at fixed mass, energy and Casimir invariants. This leads to an optimal $\rho_*(\mathbf{r},\mathbf{v},\eta) = \frac{1}{Z(\mathbf{r})}\chi(\eta)e^{-\eta(\beta\epsilon+\alpha)}$ from which we obtain the optimal coarse-grained distribution

$$\overline{f} = \frac{\int \chi(\eta)\eta e^{-\eta(\beta\epsilon+\alpha)}d\eta}{\int \chi(\eta)e^{-\eta(\beta\epsilon+\alpha)}d\eta} = f_{L.B.}(\epsilon), \tag{4}$$

where α and β are the usual Lagrange multipliers associated with the conservation of mass M and energy E while $\chi(\eta) = \exp\{-\sum_n \alpha_n\eta^n\}$ takes into account the conservation of all the Casimirs invariants I_n [5]. We note that, even if the system mixes well (ergodicity), the coarse-grained DF predicted by Lynden-Bell may *differ* from the Boltzmann distribution: $f_{L.B.}(\epsilon) \neq Ae^{-\beta\epsilon}$ (in general) due to the presence of the Casimir invariants. In the simplest case (two-levels approximation) the DF predicted by Lynden-Bell is similar to the Fermi-Dirac statistics: $f_{L.B.} = \eta_0/(1+e^{\beta\epsilon-\mu})$ [9]. More generally, the coarse-grained DF (4) is a sort of *superstatistics* [5,10] as it is expressed as a superposition of Boltzmann distributions (universal) weighted by a non-universal factor $\chi(\eta)$ depending on the initial conditions. Furthermore, like for the Beck-Cohen superstatistics, the coarse-grained DF (4) maximizes a "generalized entropy" in $\overline{f}$-space $S[\overline{f}] = -\int C(\overline{f})d\mathbf{r}d\mathbf{v}$ with $C(\overline{f}) = -\int^{\overline{f}}[(\ln\hat{\chi})']^{-1}(-x)dx$ (where $\hat{\chi}(\Phi) = \int_0^{+\infty}\chi(\eta)e^{-\eta\Phi}d\eta$) at fixed mass and energy [5]. The distribution (4) and the corresponding entropy $S[\overline{f}]$ are non-universal. However, a general prediction of the Lynden-Bell statistical theory [7] is that the QSS is a stationary solution of the Vlasov equation of the form $\overline{f} = \overline{f}(\epsilon)$ with $\overline{f}'(\epsilon) < 0$: the DF depends only on the energy and is monotonically decreasing. Furthermore, $\overline{f}(\mathbf{r},\mathbf{v}) \leq \max_{\mathbf{r},\mathbf{v}} f(\mathbf{r},\mathbf{v},t=0)$: the coarse-grained DF is bounded by the maximum value of the initial DF.

4 Incomplete violent relaxation

The Lynden-Bell DF (4) is the proper prediction of the QSS when mixing is efficient (ergodic) during violent relaxation. There are situations where the Lynden-Bell prediction works relatively well, e.g. [11]. However, it has been recognized in many other occasions [4] that mixing is *not* efficient enough to sustain the hypothesis of ergodicity on which the theory is built so that the prediction of Lynden-Bell fails in practice: $f_{QSS} \neq f_{LB}(\epsilon)$ (in general) [1]. This is particularly obvious in the case of self-gravitating systems because the Lynden-Bell entropy has no maximum at fixed (finite) mass and energy [7,9]. What can we do to account for *incomplete relaxation*?

A first possibility would be to change the form of entropy. For example, we could try to use the "generalized thermodynamics" of Tsallis. However, as explained previously, the proper density probability is $\rho(\mathbf{r}, \mathbf{v}, \eta)$ so that the proper form of Tsallis entropy in the context of Vlasov systems is [5]:

$$S_q[\rho] = -\frac{1}{q-1} \int \left[\rho^q(\mathbf{r}, \mathbf{v}, \eta) - \rho(\mathbf{r}, \mathbf{v}, \eta)\right] d\mathbf{r} d\mathbf{v} d\eta, \tag{5}$$

instead of (1). For $q = 1$ it returns the Lynden-Bell entropy (3). In this line of thought, q would measure the efficiency of mixing ($q = 1$ if the evolution is ergodic). For $q \neq 1$, the functional (5) could describe non-ergodic behaviours (incomplete mixing). However, it is not clear why all non-ergodic behaviours could be described by a simple functional such as (5). Tsallis entropy may describe a certain type of non-ergodicity (fractal or multifractal) with phase-space structures but probably not all of them. In particular, observations of elliptical galaxies in astrophysics [4] do *not* favour Tsallis form of entropy since elliptical galaxies are not *stellar polytropes* that would be the prediction based on the q-entropy [5]. Therefore, $f_{QSS} \neq f_q(\epsilon)$ (in general).

Another possibility is to keep the Lynden-Bell entropy as the most fundamental entropy of the problem but develop a *dynamical* theory of violent relaxation. Indeed, if relaxation is incomplete, we must understand why. Qualitatively, the collisionless relaxation is driven by the fluctuations of the field $\Phi(\mathbf{r}, t)$. Now, these fluctuations can vanish before the system had time to relax completely so that the system can remain frozen in a stationary state of the Vlasov equation which is not the most mixed state. By using a phenomenological Maximum Entropy Production Principle (MEPP), we have proposed in [12] to describe the out-of-equilibrium evolution of the probability density $\rho(\mathbf{r}, \mathbf{v}, \eta, t)$ by a relaxation equation of the form

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \frac{\partial \rho}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial \rho}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \cdot \left\{ D(\mathbf{r}, \mathbf{v}, t) \left[\frac{\partial \rho}{\partial \mathbf{v}} + \beta(t)(\eta - \overline{f})\rho \mathbf{v} \right] \right\}. \tag{6}$$

If the diffusion coefficient were constant, this equation would relax towards the Lynden-Bell distribution. However, it is argued in [12] that D is *not* constant. Indeed, the relaxation is driven by the fluctuations of the field Φ, itself induced by the fluctuations of f, so that the diffusion coefficient should be proportional to $\overline{\tilde{f}^2} = \overline{f^2} - \overline{f}^2$ and vanish in the regions of phase space where these fluctuations vanish. Moreover, as the system approaches (quasi)-equilibrium, the fluctuations of the field $\delta\Phi$ are less and less efficient so that the diffusion coefficient should also decay with time. For these reasons, it can become very small $D(\mathbf{r},\mathbf{v},t) \rightarrow 0$ in certain regions of phase space (where mixing is not very efficient) and for large times (as the fluctuations weaken). The diffusion coefficient can also rapidly decay with the velocity. The vanishing of the diffusion coefficient can "freeze" the system in a subdomain of phase space (bubble) and account for incomplete relaxation and non-ergodicity. The relaxation equation (6) should then tend to a distribution which is only *partially mixed* and which is usually different from the Lynden-Bell and the Tsallis distributions. However, this approach demands to solve a dynamical equation (6) -smoother than the Vlasov equation- to predict the metaequilibrium state. The idea is that, in case of incomplete relaxation (non-ergodicity), the prediction of the QSS is impossible without considering the dynamics: it depends on the "route to equilibrium".

Maybe, we have to accept that, in general, the QSS is *unpredictable* in case of incomplete relaxation. We can expect, however, that $\overline{f}_{QSS}(\mathbf{r},\mathbf{v})$ is a stable stationary solution of the Vlasov equation. We are thus led to construct stable stationary solutions of the Vlasov equation is order to *reproduce* observations. The Vlasov equation admits an infinite number of stationary solutions given by the Jeans theorem, but not all of them are stable. Of course, only stable solutions must be considered and their selection is a difficult problem. If we restrict ourselves to DF of the form $f = f(\epsilon)$ with $f'(\epsilon) < 0$ depending only on the energy (in astrophysics, such distributions characterize a subclass of *spherical* stellar systems), it is possible to provide a simple criterion of nonlinear dynamical stability. Such DF extremize a functional of the form $H[f] = -\int C(f) d\mathbf{r} d\mathbf{v}$, where C is convex, at fixed mass and energy [13]. Indeed, the first variations $\delta H - \beta\delta E - \alpha\delta M = 0$ yield a DF of the form $f = F(\beta\epsilon + \alpha)$ with $F(x) = (C')^{-1}(-x)$ montonically decreasing. If, furthermore, the DF *maximizes* this functional (at fixed E, M), then it is nonlinearly dynamically stable with respect to the Vlasov equation [13,8,14]. The intrinsic reason is that $H[f]$, $E[f]$ and $M[f]$ are individually conserved by the Vlasov equation. Therefore, if $f_0(\mathbf{r},\mathbf{v})$ is the maximum of $H[f]$ (at fixed E, M), a small perturbation $f(\mathbf{r},\mathbf{v},t)$ will remain close (in some norm) to this maximum. It is important to note that this criterion of nonlinear dynamical stability is remarkably consistent with the phenomenology [5] of violent relaxation if we view the relevant DF as the *coarse-grained* DF. Indeed, during mixing $d\overline{f}/dt \neq 0$ and the functionals $H[\overline{f}] = -\int C(\overline{f}) d\mathbf{r} d\mathbf{v}$ calculated with the coarse-grained DF increase ($-H$ decrease) in the sense that $H[\overline{f}(\mathbf{r},\mathbf{v},t)] \geq H[\overline{f}(\mathbf{r},\mathbf{v},0)]$ for $t \geq 0$ where it is assumed that initially the system is not mixed: $\overline{f}(\mathbf{r},\mathbf{v},0) = f(\mathbf{r},\mathbf{v},0)$. Because of this property similar to the Boltzmann H-theorem in kinetic theory [1], $H[\overline{f}]$ are called generalized

[1] Note that the Vlasov equation does not single out a unique H-function contrary

H-functions [13]. By constrast, $E[\overline{f}]$ and $M[\overline{f}]$ are approximately conserved. Therefore, this generalized *selective decay principle* [5] (decrease of $-H[\overline{f}]$ at fixed E, M) due to phase mixing and coarse-graining can explain *how* $\overline{f}$ can possibly reach a maximum of H at fixed mass and energy (while $H[f]$ is rigorously conserved on the fine-grained scale). After mixing $d\overline{f}/dt = 0$ and the functionals $H[\overline{f}]$ (as well as $E[\overline{f}]$ and $M[\overline{f}]$) are conserved by the coarse-grained flow. Therefore, if $\overline{f}$ has reached (as a result of mixing) a maximum of H, it will be nonlinearly dynamically stable with respect to coarse-grained perturbations (after mixing) in virtue of the above-mentioned stability result.

In reality, the problem is more complicated because the system can converge toward a stationary solution of the Vlasov equation which does not depend on the energy ϵ alone, and thus which does not maximize an H-function at fixed E, M. For example, the velocity distribution of stars in elliptical galaxies is anisotropic and depends on the angular momentum $J = |\mathbf{r} \times \mathbf{v}|$ in addition to energy ϵ. Furthermore, real stellar systems are in general not spherically symmetric so their DF does not only depend on ϵ and J. Therefore, more general stationary solutions of the Vlasov equation must be constructed in consistency with the Jeans theorem. Elliptical galaxies are well relaxed (in the sense of Lynden-Bell) in their inner region (leading to an isotropic isothermal core [2] with density profile $\sim r^{-2}$) while they possess radially anisotropic envelopes (with density profile $\sim r^{-4}$). Stiavelli & Bertin [4] introduced an $f^{(\infty)}$ model of the form $f^{(\infty)} = A(-\epsilon)^{3/2} e^{-a\epsilon - cJ^2/2}$ for $\epsilon \le 0$ (and $f = 0$ otherwise) based on the possibility that the *a priori* probabilities of microstates are not equal due to kinetic constraints. This model reproduces many properties of ellipticals but it has the undesired feature of being "too isotropic". Then, they introduced another model based on a modification of the Lynden-Bell statistical theory. They considered the maximization of the Boltzmann entropy (in Lynden-Bell's sense) at fixed mass, energy *and* a third global quantity $Q = \int J^\nu |\epsilon|^{-3\nu/4} f d\mathbf{r} d\mathbf{v}$ which is argued to be approximately conserved during violent relaxation. This variational principle results in a family of $f^{(\nu)}$ models $f^{(\nu)} = A \exp[-a\epsilon - d\left(J^2/|\epsilon|^{3/2}\right)^{\nu/2}]$. These models are able to fit products of N-body simulations over nine orders of magnitude in density and to reproduce the de Vaucouleur's $R^{1/4}$ law (or more general $R^{1/n}$ laws) of ellipticals. The introduction of additional constraints in the variational principle could be a way to take into account effects of incomplete violent relaxation.

We note finally that the Vlasov equation can have a very complicated, non-

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to the Boltzmann equation (the above inequality is true for all H-functions) and the time evolution of the H-functions is not necessarily monotonic (nothing is implied concerning the relative values of $H(t)$ and $H(t')$ for $t, t' > 0$).

[2] One success of Lynden-Bell's theory of violent relaxation is precisely to explain the isothermal cores of elliptical galaxies without recourse to "collisions" whose effect manifests itself on a much longer timescale.

ergodic, dynamics. For example, in the gravitational 1D Vlasov-Poisson system, phase-space holes which block the relaxation towards the Lynden-Bell distribution (incomplete relaxation) have been observed [15]. In that case, the system does not even relax towards a stationary state of the Vlasov equation but develops everlasting oscillations. Rapisarda & Pluchino [16] have also observed transient phase-space structures in their N-body simulations of the HMF model and they have proposed an interesting analogy with glassy dynamics. These results are not necessarily in contradiction with the Vlasov equation (as they beleive), even for the inhomogeneous situations that they consider. It would be of interest to check whether these phase-space structures can also be obtained by solving the Vlasov equation for the HMF model.

5 Summary and conclusion

Non-Boltzmannian distributions appear in the study of Hamiltonian systems with long-range interactions. For these systems, the collisional relaxation time towards statistical equilibrium (Boltzmann distribution) is huge because it increases as a power of the number of particles N. Therefore, the evolution of the system is described by the Vlasov equation on a very long timescale [1]. Now, because of phase mixing and violent relaxation, the Vlasov equation can spontaneously lead to the formation of coherent structures: galaxies in astrophysics, jets and vortices in hydrodynamics, clusters in the HMF model, bars in disk galaxies... These QSS are (nonlinearly) dynamically stable stationary solutions of the Vlasov equation which are not necessarily described by the Boltzmann distribution. Indeed, the Vlasov equation admits an infinite number of stationary solutions and the system can be trapped in one of them. In general, the QSS is *non-universal* as it depends: (i) on the *detailed* structure of the initial conditions (through the Casimirs, in addition to mass and energy) and (ii) on the efficiency of mixing (ergodicity). In this Vlasov context, the Tsallis distributions are *particular* stationary solutions of the Vlasov equation which correspond to what are called *stellar polytropes* in astrophysics [5,14]. In addition, the Tsallis functional $S_q[\overline{f}]$ is a *particular* H-function [13], not an entropy (which would be a functional $S_q[\rho]$). Its maximization at fixed mass and energy yields a criterion of nonlinear dynamical stability with respect to the Vlasov equation [8,13,14], not a criterion of generalized thermodynamical stability. A formal *thermodynamical analogy* [14] can however be developed to investigate the nonlinear dynamical stability problem.

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