Renormalizing the chaotic dynamics of motile particles in fractal porous media.

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Natural porous media often display a fractal character which is manifest through a fractal Eulerian velocity or conductivity field. By assuming the fractal Eulerian velocity gives rise to a fractal Lagrangian drift velocity, we may model the motile particle paths in the spirit of a nested hierarchy of stochastic ordinary differential equations (SODEs) with stochastic drift. If we further assume that the motile particles have a preferred direction of travel (e.g., chemotaxis) and/or that the porous media is anisotropic, it is possible to represent both the drift and diffusion as operator stable processes. Further, if the hydraulic conductivity is statistically homogeneous, then it is not unreasonable to assume it imparts on a particle a Lagrangian drift velocity with stationary increments. With these assumptions at hand, we develop a rigorous renormalization procedure and derive the renormalized Fokker-Planck equations for particle trajectories over the hierarchy. As an illustrative example of the general procedure we employ Levy motions to model both the motility at the microscale and drift at the mesoscale. On the microscale (pore scale) the motile particle is modeled as an operator stable Levy process with stationary, ergodic Markov drift velocity. The micro to meso and meso to macro scale homogenization is handled with generalized central limit theorems which can be shown equivalent to a renormalization group approach. On the mesoscale, to account for the fractal medium, the Lagrangian drift is modeled as an operator stable Levy process, however, on this scale the diffusion is the asymptotic limit of the total microscale process. At the macroscale the process is the asymptotic limit of the total mesoscale process. The Fokker-Plank equations at each scale possess fractional spatial derivatives, the order of which can be obtained via particle tracking experiments and the finite-size Lyapunov exponent (FSLE). The FSLE is the exponential rate at which two particles separate from a distance \( r \) to \( ar^\alpha \) (\( \alpha > 1 \)) and provides a measure of the dispersive mixing in chaotic systems. We can show analytically that for alpha-stable Levy processes, the FSLE is proportional to the diffusion coefficient and its log is proportional to the negative of the stability constant.