Probability analysis of nonlinear dynamical systems driven by Ornstein-Uhlenbeck process

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The advantage of the white noise approximation as a simulation of the impact of a random environment on the considered nonlinear dynamical system is that it allows one to apply the powerful apparatus of the theory of Markovian processes and thereby obtain exact analytical results. Although the white noise approximation is a very convenient theoretical technique, in physical applications the correlation time of real noise, however small, is strictly non-zero. Any attempts to theoretically consider nonlinear systems under the influence of real external noise are faced with the fact that the evolution of the system in time ceases to be Markovian. At the same time, the noise generated by the environment can be considered for simplicity as a Markovian and, representing the combined effect of the action of many weakly coupled factors, it is distributed according to the Gaussian law by the central limit theorem. These properties of external fluctuations (ergodicity, Markovianity, and Gaussianity) limit the wide choice of models to only one possibility. Indeed, a regular stationary Gaussian Markovian process is an Ornstein-Uhlenbeck process. Thus, it can be concluded that for most applications, the Ornstein-Uhlenbeck process is a suitable model for external colored noise.

In this paper a new approximate method for finding the steady-state probability distribution of a nonlinear dynamical system described by first-order differential equation with additive colored Gaussian noise (Ornstein-Uhlenbeck process) is proposed. An expansion of a stationary probability density function into a power series in a small parameter, the noise correlation time, up to second-order terms, has been obtained for the first time. The procedure is based on solving an infinite chain of differential equations for conditional moments, obtained from Fokker-Planck equation for the joint probability distribution. Based on the power series expansion obtained, the applicability of the previously proposed approximations is estimated. As shown, the previously proposed steepest-descent approximation gives an incorrect result already in the first order term of expansion, while the unified colored noise approximation and the functional-calculus approach only in the second order. At the end, an exact probability analysis of the Gompertz model of tumor cell growth with fluctuations in the form of the Ornstein-Uhlenbeck process in the growth rate is carried out. Along with the steady-state probability density function of tumor volume, the evolution of the conditional probability distribution in time is precisely found. The dependence of the stationary probability distribution on the correlation time and intensity of external fluctuations is studied in detail. As was shown, the form of this probability density function is unimodal, and the most probable value of the tumor volume, as well as its mean value, increases with increasing the correlation time of noise, but decreases with increasing its intensity.