First-passage time below the diagonal for the Brownian maximum

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For $t \geq 0$, let S_t be the running maximum of a standard Brownian motion B_t and let $T_m := inf\{t;\ mS_t < t\}, m > 0$. In this talk I will discuss the calculation of the joint distribution of T_m and B_{T_m}. The motivation for this work comes from a mathematical model for animal foraging. The toy-model which we have in mind was imagined by P. Krapivsky. It deals with the simplified, stylized case of an animal foraging in a one dimensional space. The animal's initial position coincides with the origin, and we model its position as time t elapses by a standard Brownian motion (B_t), $t \geq 0$. For the sake of simplicity, we suppose that the forager's metabolism is basic: to survive, the animal needs one unit of food per unit time, and it may stockpile any extra supply for future use, without any upper limit on the size of the stock nor any expiry date for the consumption thereof. Assume that only half of the space (say, the positive half-line) is initially filled with one unit of food per unit length, and that there is no replenishment. Thus, if the animal's motion is modelled by a Brownian motion, after a time t the forager has absorbed an amount of food equal to S_t, its maximal displacement in the positive direction. For the forager to survive up to a time t, it should be the case that, at every time $s \leq t$, the amount of food it had absorbed was not less than s. In other terms, the probability that the forager survives up to a time t is given by the probability that $S_s \geq s$ for all s in,$[0, t]$. Equivalently, this is the probability that the first (downward) hitting time T of the maximum process S_t on the diagonal barrier occurs after t-- which relates directly to the problem stated at the beginning of this abstract. I will show a path transformation that allows to calculate the joint distribution of T_m and B_{T_m}. Time-permitting, I will also present results for Brownian motion with drift.

References

[1] J. Randon-Furling, P. Salminen, P. Vallois, On a first hit distribution of the running maximum of Brownian motion, Stochastic Processes and their Applications, 150, 1204-1221 (2022).