Diffusion as a signature of chaos in classical and quantum systems

David Campbell[1], Nachiket Karve, Nathan Rose
[1]Boston University, Brookline, United States

Sensitive dependence on initial conditions (SDIC) is a defining feature of chaos in classical dynamical systems and is usually quantified via Lyapunov exponent. However, the Lyapunov exponent is a short-time probe and can fail to capture the long-time behavior of the system. An example is the inner solar system, where previous studies have estimated the Lyapunov time (the inverse of the Lyapunov exponent) to be five million years. This time-scale is much shorter than the estimated age of the solar system, which is about 5 billion years ! The Lyapunov exponent's failure to capture long-time dynamics is a consequence of its definition relying on infinitesimally small perturbations. But as any finite separation eventually becomes large, nonlinear effects start to affect the distance between two trajectories. Therefore, we will adopt a statistical view of chaos through the concept of ”weak mixing”. As the name suggests, a localized ensemble of initial conditions in a mixing system eventually spreads throughout the entire phase space. Such systems are irreversible and information about the initial state is lost at later times. Mixing systems also exhibit SDIC, since any tow initial states are eventually separate from each other. Importantly, the property of mixing implies SDIC at later times. We shall refer to such systems as being ”chaotic in the mixing sense.” In quantum systems, in the other hand, there is no direct analogue of SDIC, since unitary time evolution is a quasi-perdiodic object. Recent studies had established that one can define quantum chaos via sensitivity to adiabatic deformations and can probe this senstiivity via the Adiabatic Gauge Potential (AGP). We will refer to quantum and classical systems that are sensitive to adiabatic deformations as being ”chaotic in the AGP sense.” In the present work we reconcile these two notions of chaos–in the mixing sense and in the AGP sense. We generalize the formalism that probes chaos via the diffusion of time-integrated observables and show that this probe correctly classifies mixing systems as chaotic. Further, this probe allow us to classify both transient and asymptotic dynamics into four distinct classes, which we refer to as dissipative, regular, strongly chaotic, and weakly chaotic. Regular and dissipative dynamics are characterized by the absence of diffusion in the time-integrated observables, whereas strong and weak chaos correspond to normal and anomalous diffusion, respectively. We show that absolutely decaying correlations–a feature of weak mixing–cause this diffusion and that the rate of decay determines the strength of the chaos. We show that simple examples of discrete-time maps demonstrate the efficacy of this method.