Statistical mechanics of Hamiltonian curl-force systems

Omer Chor[1], Aljaž Godec[2], Oren Raz[1]

[1]Weizmann Institute Of Science, Rehovot, Israel, [2]University of Freiburg, Freiburg, Germany

We investigate the statistical mechanics of a particle subject to Hamiltonian curl forces—reversible, velocity-independent, non-conservative forces with non-vanishing curl that can be described by a Hamiltonian with an anisotropic kinetic energy term [1]. We demonstrate that these systems generally do not relax to a Boltzmann distribution when coupled to a heat bath; consequently, the Hamiltonian structure cannot be easily exploited to determine their nonequilibrium steady states.

We analyze the Klein–Kramers equation describing the underdamped stochastic evolution of such a system. Our central result is that, under certain constraints on the kinetic energy anisotropy, a canonical transformation maps the curl-force problem onto an equivalent stochastic system characterized by a standard isotropic kinetic energy but coupled simultaneously to multiple heat baths at different temperatures. In this transformed representation, the system is no longer driven by a non-conservative curl force; instead, its nonequilibrium nature arises solely from anisotropic thermal noise applied on different degrees of freedom. This establishes a direct formal correspondence between two conceptually distinct mechanisms for driving a system out of equilibrium: deterministic non-conservative forcing and coupling to multiple heat reservoirs.

This mapping is applied to two representative examples. First, in the case of quadratic potentials, the transformed system reduces to the Brownian gyrator [2], a minimal model for a stochastic heat engine. As an exactly solvable Ornstein–Uhlenbeck process, this model enables a direct comparison of work, heat dissipation, and entropy production rates across both representations. Second, we examine a gyrator confined by a ring potential [3], where the curl of the force field in the transformed system exhibits a quadrupolar structure that accurately predicts the local circulation in the two-temperature description. In both cases, the curl-force representation provides an intuitive mechanical interpretation of the origin and direction of the rotational currents characteristic of their nonequilibrium steady states.

References:

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