Fractal Complex Networks: From Local to Global Scaling Relations

Piotr Fronczak[1], Agata Fronczak[1], Maciej J. Mrowiński[1], Michał Łepek[1], Mateusz J. Samsel[1], Kordian Makulski[1]

[1]Warsaw University Of Technology, Warsaw, Poland

Fractality in complex networks has been widely observed across a broad range of systems, including technological, biological, and social networks. Despite its ubiquity, the fundamental origin of network fractality and its structural implications remain only partially understood [1]. In this presentation, we introduce a refined scaling theory of fractal complex networks, focusing on the fundamental relations between microscopic (local) and macroscopic (global) structural properties [2].

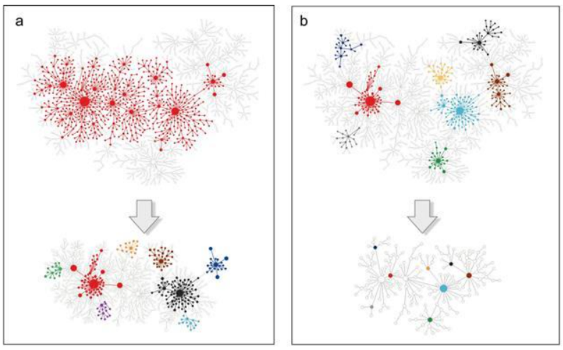

Fig. Schematic illustration of the idea of geometric self-similarity in complex networks. Part (a) shows that the network can be subdivided into parts-boxes of a given diameter-each of which is (at least approximately) a reduced-size copy of the entire network. One such box, marked in red, is extracted from the original network and treated as a new network (shown below). It is divided again into new smaller boxes, some of which are marked with different colours. Part (b) illustrates renormalization procedure applied to the same network as in part a. The top original network is divided into boxes of a fixed diameter, some of which are marked with different colours. In the new network after renormalization (shown below), these boxes are replaced by nodes with the corresponding colours. In both approaches, the macroscopic and microscopic characteristics of the network after renormalization are similar to those of the original network.

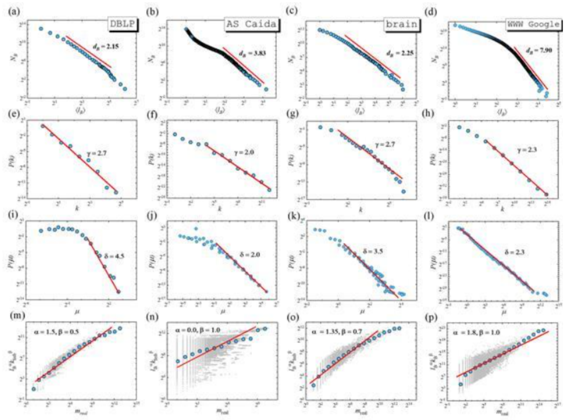

Fig. 2. Three scaling relations for four different complex networks (top three rows) and a verification of the scaling theory (fourth row; the red line indicates the expected proportionality).

Our framework establishes a direct connection between three microscopic exponents—degree, geometric, and topological scaling exponents—and three macroscopic ones: the fractal dimension, as well as the exponents governing the power-law distributions of node degrees and box masses. We demonstrate that these quantities are not independent, but are linked through a set of scaling relations derived from the network invariance under renormalization. As a consequence, knowledge of one subset of exponents uniquely determines the others, providing a powerful and predictive tool for multiscale network characterization.

We validate the proposed theory using several real-world fractal networks, including the Internet at the level of autonomous systems, subsets of the World Wide Web, and scientific collaboration networks, as well as synthetic models. Across all cases, we observe strong agreement between theoretical predictions and empirical data.

In addition, we introduce a novel computational approach, recently proposed in the literature [3], that addresses the limitations of traditional box-covering algorithms. Standard methods, such as Greedy Coloring, are often computationally expensive and sensitive to the choice of box size. In contrast, the Fixed Number of Boxes algorithm provides a more efficient and robust alternative. Importantly, this method is not purely heuristic: it is directly motivated by the scaling relations derived within our theoretical framework, ensuring consistency between analytical predictions and numerical implementation. By identifying boxes through local hubs and fixing their number rather than their diameter, the algorithm yields more accurate estimates of the fractal dimension in both synthetic and empirical networks.

A key result of this integrated approach is the identification of a fundamental link between the macroscopic fractal dimension and the microscopic scaling of hub connectivity. In particular, we show that the growth of the maximum degree within a box, governed by its own scaling exponent, acts as a bridge between local renormalization dynamics and global network topology. This provides a clear explanation for the persistence of self-similar hub structures across scales.

References:

[1] C. Song, S. Havlin, H.A. Makse, Origins of fractality in the growth of complex networks, Nat. Phys. 2, 275 (2006)

[2] A. Fronczak, P. Fronczak, M.J. Samsel, et al., Scaling theory of fractal complex networks, Sci. Rep. 14, 9079 (2024).

[3] M. Łepek, K. Makulski, A. Fronczak, P. Fronczak, Beyond traditional box-covering, Chaos, Solitons Fractals, 199, 116908 (2025)