Karhunen-Loeve Expansion of the Brownian Oscillator Kernel

Nikolaos Paraskakis[1], Dionissios T. Hristopulos[1]

[1]Technical University Of Crete, Chania, Greece

We present semi-explicit analytical relations for the Karhunen-Loève (KL) expansion of the linear damped harmonic oscillator driven by white-noise excitation – also known as Brownian oscillator (BO). Central to statistical physics and stochastic dynamics, this model captures the interplay between energy dissipation and storage in noisy oscillatory systems and arises in mechanical, electrical, and biological systems, as well as stochastic transport phenomena.

The KL expansion provides an optimal orthogonal decomposition of stochastic processes into uncorrelated modes over a specific time interval. The KL expansion can be viewed as the continuous-time analogue of Principal Component Analysis (PCA). For Gaussian processes, it yields statistically independent coefficients, facilitating entropy evaluation and rendering rate-distortion formulations tractable.

Analytical KL expansions are available only for a limited class of covariance kernels. Closed-form solutions exist for the Wiener and Ornstein-Uhlenbeck processes, where the eigenvalue problems reduce to tractable forms within Sturm-Liouville theory. For more general stochastic systems, and in particular for second-order dynamical models such as the Brownian oscillator, explicit KL representations are typically unavailable.

For the BO, damping and oscillatory dynamics produce a nontrivial spectral structure that varies across regimes. The qualitative behavior of eigenfunctions and the eigenvalue decay rate depend on whether the system is underdamped, critically damped, or overdamped, leading to distinct modal structures and truncation efficiencies.

We derive the KL expansion across all regimes by casting the eigenvalue problem as a boundary-value problem governed by a fourth-order linear ordinary differential equation (ODE). The resulting solutions exhibit oscillatory, exponential, or mixed-type behavior, depending on the damping regime. Analysis of the characteristic equation yields a complete classification of admissible solution families and semi-explicit representations for eigenvalues and eigenfunctions. The formulation directly links damping ratio and natural frequency to spectral properties, providing physical interpretation alongside the mathematical derivation.

Numerical results show rapid spectral decay in the underdamped regime, enabling compact low-rank representations, while slower decay in the overdamped regime reflects increased modal complexity. We address the conditioning of the resulting transcendental eigenvalue equations and introduce numerically stable formulations across all regimes, including cases with exponentially growing components.

The resulting BO KL basis enables efficient low-rank representations of stochastic processes, supporting compression, filtering, and denoising, while also enabling hybrid spectral machine learning frameworks combining analytical KL modes with data-driven learning. The framework provides a unified spectral characterization of the BO, bridging stochastic dynamics, spectral analysis, and information-theoretic perspectives. Further research will focus on low-rank Gaussian process regression for oscillatory time series such as sunspot activity, where the BO kernel provides a natural modeling framework.

Acknowledgment This research was funded by the research project SLIMNETS, which is implemented in the framework of H.F.R.I call ”3rd Call for H.F.R.I.'s Research Projects to Support Faculty Members & Researchers” (H.F.R.I. Project Number: 25024).