Exact Finite-N Partition Functions in Matrix and Tensor Quantum Mechanics and Negative Specific Heat

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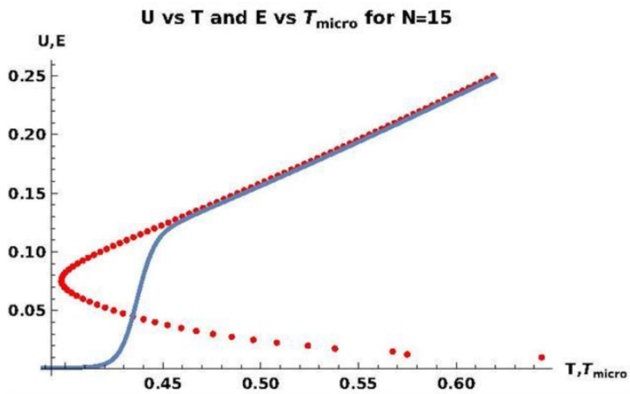
Gauged harmonic oscillator quantum mechanical models with matrix or tensor variables of size scaling with N admit representation theoretic expressions for the micro-canonical partition functions. The gauge symmetry in some cases of interest is a continuous group e.g. the unitary $U(N)$ and in other cases it is a finite group, e.g. the symmetric group S_N. The representation theoretic expressions involve, for example, characters and Kronecker coefficients of symmetric groups. The talk describes families of models with negative heat capacities in the micro-canonical ensemble and ensemble inequivalence controlled by N, with ensemble equivalence being recovered at high energies.

In these models, the micro-canonical degeneracies have near-exponential or super-exponential asymptotic forms for energies which are large but less than a characteristic scale $f(N)$ in the large N limit. For energies much greater than $f(N)$ they approach the thermodynamics of decoupled oscillators. There is a characteristic caloric fold in the energy-temperature curve in the micro-canonical ensemble, with the heat capacity changing sign and the thermodynamic behaviour approaching that of decoupled harmonic oscillators at high energy. Depending on the model studied, $f(N)$ is order N^2 or $N \log N$.



The canonical partition functions show only positive heat capacities, in line with standard general arguments. There is a Hagedorn transition in the region of the caloric fold. The ensemble inequivalence is visible for energies small compared to $f(N)$ while ensemble equivalence is recovered at energies much higher than $f(N)$. Exact representation theoretic formulae for the micro-canonical ensemble, and in some cases, the canonical ensemble, allow quantitative comparisons between the two ensembles in the transition region.

In a 2-matrix example with $U(N)$ gauge symmetry, which provides a well-known toy model for thermodynamics in the AdS/CFT correspondence, there is an additional gauging which leads to the caloric fold characteristic, which suggests an interpretation in terms of the transition between small and large AdS black hole identified by Hawking and Page.

The 3-index complex tensor model with $U(N)$ gauge symmetry has a zero temperature Hagedorn transition, as do matrix and tensor models with S_N gauge symmetry. The S_N models admit useful analytic expressions for the canonical partition functions for general values of N. These involve sums over partitions of N, where the summands take product forms depending on least common multiples of multiple parts of the partition. These analytic expressions allow computational study of the Hagedorn transition region for values large values of N approaching a hundred.

The canonical partition functions for the cases of interest with continuous symmetries, e.g. $U(N)$, are known to be rational functions of $x = e^{-1/T}$, but the forms of the numerator and denominator are not known beyond $N = 7$. These rational forms encode the generators and relations for multi-matrix or tensor gauge invariants.

The key characteristics of the comparison between canonical and micro-canonical ensemble admit some plausible conjectures based on the available data, and form interesting open problems at the intersection of thermodynamics, quantum gravity and representation theory.

The talk will be based on the papers:

https://arxiv.org/abs/2506.18813 ; JHEP12(2024)161 ; JHEP04(2024)080 ; JHEP07(2024)152