

The hyperlink representation of entanglement and the inclusion-exclusion principle

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Information and entropy are central concepts in statistical mechanics. Concretely, the mutual information between two parties is widely used to shed light into data obtained in many different scientific fields [1]. We may therefore ask about the amount of information shared between three or more parties which is not contained in any of the pairs. Higher-order information theory is a well developed branch of mathematics, but its application into physics is severely delayed. For example, concepts such as redundancy and synergy should play a more relevant role in our statistical analysis. In simple terms, three parties are said to show redundancy when knowing two of them makes the third unnecessary, while they show synergy when the full information about the system is only achieved when we know all of them, and no two of them will suffice [2]. Interestingly, synergy is strongly related to the quantum concept of monogamy [3].

On the other hand, our group showed in 2020 that the entanglement entropy (EE) of any bipartition of a pure state can be approximately expressed as a sum of entanglement links (ELs) [4]. In this work, we introduce their exact extension, i.e. the entanglement hyperlinks (EHLs), a type of higher-order mutual informations defined through the inclusion-exclusion principle, each of which captures contributions to the multipartite entanglement that are not reducible to lower-order terms [5].

We show that any EHL crossing a factorized partition must vanish, and that the EHLs between any set of blocks can be expressed as a sum of all the EHLs that join all of them. This last result allows us to provide an exact representation of the EE of any block of a pure state, from the sum of the EHLs which cross its boundary. In order to illustrate their rich structure, we discuss some explicit numerical examples using local ground states. The EHLs thus provide a remarkable tool to characterize multipartite entanglement in quantum information theory and quantum many-body physics.

References:

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