Analytical solution of overdamped Brownian motion in a periodic step potential

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Overdamped Brownian motion in periodic potentials is a paradigmatic model for transport and relaxation. Applications range from transport in optical lattices and Josephson junctions to diffusion in biological and soft-matter systems [1]. Previous studies have focused on universally symmetric step landscapes, leaving the dynamics in asymmetric unit cells unexplored [2]. Here, we present a complete analytical solution to the problem of overdamped Brownian motion in a one-dimensional periodic step potential with an arbitrary (asymmetric) barrier-well ratio.

We solve the Smoluchowski equation using a separation-of-variables and Bloch ansatz, reducing the problem to an eigenvalue problem of the Smoluchowski operator in a single unit cell of the potential. We obtain the complete set of Bloch modes and decay rates, yielding parameter-explicit expressions for the probability density, the intermediate scattering function (ISF), and the mean-squared displacement (MSD) across all times and wave numbers. We formulate the ISF in a generalized matrix form, resolving couplings between lattice-modulated density modes. In addition to the conventional diagonal elements, we explicitly analyze off-diagonal components, establish their symmetry properties via a centered representation, and show that all contributions share a universal terminal decay governed by the slowest relaxation band. Importantly, the generalized ISF we derive is directly accessible in experiments through dynamic light scattering, particle tracking, or dynamic differential microscopy, providing a direct link between theoretical predictions and observed particle dynamics [3].

For rational barrier-well ratios, the characteristic condition can be expressed as a Chebyshev polynomial problem. This allows us to determine precisely which unit cell geometries can have closed analytical solutions to the eigenvalue problem and in general reduces the issue of finding an infinite number of eigenvalues to a finite number within a given interval. Although the eigenvalues and eigenfunctions of the Smoluchowski operator can generally be computed numerically for arbitrary barrier-well ratios, our Chebyshev-polynomial approach for rational ratios provides exact analytical solutions, enabling precise exploration of the full spectrum and dynamics.

To validate our analytical framework, we performed Brownian dynamics simulations, which confirmed excellent agreement for the probability density, ISF, and MSD at all times. Across all results, geometry and barrier height influence the dynamics and the spectral structure, relaxation rates, and effective long-time transport, providing a transparent bridge from microscopic design variables to macroscopic observables.

References:

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