

Hagedorn phase transition in self-avoiding walks and optimal paths

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A self-avoiding walk (SAW) is a path in a graph that never visits the same vertex twice. SAWs are characterized by their geometry and, despite their simple definition, they remain difficult to analyze rigorously and have a wide range of applications spanning different fields.

The introduction of disorder in the graph by assigning random edge weights leads to the problem of optimal paths in disordered media. Paths are now characterized by both their geometry and their total weight, defined as the sum of the edge weights along the path. The optimal path between two fixed endpoints is the one that minimizes the total weight. Optimal paths are closely related to different optimization problems and play a central role in numerous scientific and technological applications.

Despite the notable attention devoted to SAWs and optimal paths over the last decades, their analysis at the ensemble level remains less explored. Here, we consider an ensemble of all self-avoiding trajectories with fixed endpoints and variable length, with each path weighted by its Boltzmann factor. In the zero-disorder limit, this reduces to an ensemble of SAWs while, away from the zero-disorder limit and at the zero-temperature limit, it recovers the optimal path problem. Our model allows a complete statistical characterization and provides a unified framework for SAWs and optimal paths within a single ensemble.

Furthermore, we identify a Hagedorn phase transition in our ensemble. Hagedorn phase transitions, well known in quantum chromodynamics (QCD) and string theory, are characterized by an exponentially growing density of states (DOS) and by a critical (Hagedorn) temperature above which the canonical partition function diverges. This divergence, which may be misinterpreted as an absolute maximum temperature, is now understood as the onset of a new phase in which new degrees of freedom must be created to absorb the large amount of incoming energy resulting from the DOS. In QCD, this occurs through the creation of quark-antiquark pairs, which leads to the color deconfinement and the quark-gluon plasma phase. In string theory, increasingly long strings develop more and more excited oscillatory modes, yielding a new phase dominated by extremely long and excited strings. We show how a Hagedorn temperature and an exponentially growing DOS naturally emerge from our model, and we conclude by unveiling the similarities between our model and those discussed in QCD and string theory.

