

Synchronization in the Complexified Kuramoto model

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In this paper, we consider an N -oscillators complexified Kuramoto model. We first observe that there are solutions exhibiting finite-time blow-up behavior in all coupling regimes. When the coupling strength is large $\lambda > \lambda_c$, sufficient conditions for various types of synchronization are established for general systems of $N \geq 2$. On the other hand, we analyze the case when the coupling strength is weak. For $N = 2$ with coupling below λ_c , our complex-analytic approach not only recovers the periodic orbits reported by Thumler–Srinivas–Schroder–Timme, but also provides for the first time their exact period, confirming full phase locking. Furthermore, for the critical case $\lambda = \lambda_c$, we find that the complexified Kuramoto system admits homoclinic orbits. These phenomena significantly differentiate the complexified Kuramoto model from the real Kuramoto system, as synchronization never occurs when $\lambda < \lambda_c$ in the latter. For $N = 3$, we demonstrate that if the natural frequencies are in arithmetic progression, non-trivial synchronization states can be achieved for certain initial conditions even when the coupling strength is weak. In particular, we characterize the critical coupling strength, such that a semistable equilibrium point in the real Kuramoto model bifurcates into a pair of stable and unstable equilibria, marking a new phenomenon in complexified Kuramoto models.