

Designing topological cluster synchronization patterns with the Dirac operator

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Designing and controlling stable cluster synchronization patterns is a central problem in the study of nonlinear dynamics on complex networks, with great relevance to a wide variety of systems, including chemical oscillators and brain networks. Traditionally, this phenomenon has been studied within node-based frameworks, where dynamical variables are associated only with nodes. Within these frameworks, cluster synchronization defines a state in which different clusters of nodes in a network oscillate at different frequencies. However, these frameworks are restrictive when it comes to systems whose dynamics are more naturally modelled by dynamical variables associated with links rather than nodes, such as synaptic signals in the brain and fluxes in biological transport networks. Consequently, the study of the collective behaviour of dynamical variables associated with higher-dimensional topological structures in simple and generalized networks—such as links, triangles, and tetrahedra—has received increasing attention [1]. These generalized dynamical variables are known as topological signals, and they naturally extend the challenge of designing and controlling stable cluster synchronization patterns to complex systems that support them.

In this work [2], we leverage tools from algebraic topology to develop the Dirac-Equation Synchronization Dynamics (DESD) model, a Kuramoto-like model that describes the synchronization dynamics of topological signals defined on nodes and links. The DESD model is formulated through the topological Dirac operator, which defines the topological Dirac equation [3] and naturally couples node- and edge-level dynamics within a unified framework. By modulating the ground state of the free energy associated with the dynamical system, our model enables the targeting of specific topological synchronization patterns at steady state, given sufficiently large coupling strength. These patterns align with eigenstates of the topological Dirac equation. Importantly, some of these eigenstates induce multiple stable topological cluster synchronization patterns, in which clusters of nodes and bundles of links oscillate at distinct frequencies.

Using linear stability analysis, we provide a theoretical framework for predicting the stability of a given targeted topological synchronized state in the large-network limit. Specifically, we show that stability depends on the scaling of the spectral density of the topological Dirac operator around the eigenvalue associated with the eigenstate to which the system is aligned. Furthermore, we present numerical simulations demonstrating stable topological cluster synchronization patterns in stochastic block models and on real connectome data.

References:

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